Improper Integrals

Last class we talked about bounded intervals of bounded functions. Now we generalise to unbounded intervals. and unbounded functions.

1. Remark: $[a, +\infty] = \bigcup_{b>1} [1, b]$.

2. Definition: Assume f is integrable. Then $\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx$.

LHS only makes sense if RHS has a limit. In that case we say that $\int_a^{+\infty} f(x)dx$ exists/converges.

2.1. Example: $\int_a^{+\infty} x^\alpha dx$. For what α does our integral exist? If it exists, what's its value?

$$\int_1^b x^\alpha \, dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1}\Big|_1^b = \frac{1}{\alpha+1}(b^{\alpha+1} - 1), & \alpha \neq -1, \\ \ln x|_1^b = \ln b, & \alpha = -1. \end{cases}$$

$$\Downarrow$$

$$\int_1^b x^\alpha \, dx = \begin{cases} \dfrac{1 - b^{\alpha+1}}{-(\alpha+1)}, & \alpha < -1, \\ \ln b, & \alpha = -1, \\ \dfrac{b^{\alpha+1} - 1}{\alpha+1}, & \alpha > -1. \end{cases}$$

$$\Downarrow$$

$$\int_1^\infty x^\alpha \, dx = \begin{cases} -\dfrac{1}{\alpha+1}, & \alpha < -1, \\ \text{diees}, & \alpha \geq -1. \end{cases}$$

2.2. Remark: $x^\alpha \to 0$ **when** $x \to +\infty$**. To make the improper exist** $x^\alpha \to 0$ **must be "fast" when** $x \to +\infty$ **(e.g. faster than** x^{-12}**)**

2.3. Remark: Whether $\int_a^{+\infty} f(x)dx$ **converges or not does not depend on** a**.**

3. Definition: Assume f is integrable over any $[b, a]$. Suppose $f : [c, a] \to \mathbb{R}$. We define $\int_c^a f(x)dx = \lim_{b\to c^+} \int_b^a f(x)dx$.

Again, LHS only makes sense if RHS has a limit. In that case we say that $\int_c^a f(x)dx$ exists/converges.

3.4. Example: $\int_0^1 x^\alpha dx$**. For what** α **does our integral exist? If it exists, what's its value?**

$$\int_b^1 x^\alpha\,dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1}\Big|_b^1 = \frac{1}{\alpha+1}(1-b^{\alpha+1}), & \alpha \neq -1, \\ \ln x|_b^1 = -\ln b, & \alpha = -1. \end{cases}$$

$$\Downarrow$$

$$\int_b^1 x^\alpha\,dx = \begin{cases} \dfrac{1-b^{\alpha+1}}{\alpha+1}, & \alpha > -1, \\ -\ln b, & \alpha = -1, \\ \dfrac{1-b^{\alpha+1}}{\alpha+1}, & \alpha < -1. \end{cases}$$

$$\Downarrow$$

$$\int_0^1 x^\alpha\,dx = \begin{cases} \dfrac{1}{\alpha+1}, & \alpha > -1, \\ \text{diees}, & \alpha \leq -1. \end{cases}$$

3.5. Remark: x^{-12} **goes to** $+\infty$ **slower than** x^{-2}**.**

4. Example: $\int_1^{+\infty} \frac{\sin x}{x^2} dx$.

It's hard to determine $\int_1^b \frac{\sin x}{x^2} dx$ with a closed form with b.

5. Criterion 1: If $|f(x)| \leq g(x)$ for large x then $\int_a^{+\infty} g(x)dx$ exists $\implies$ $\int_a^{+\infty} f(x)dx$ exists.

Sketch of proof: We want to show that $\lim_{b\to+\infty} \int_a^b f(x)\,dx$ exists. By Cauchy's criterion, this is equivalent to saying that $\forall \epsilon > 0,\ > 0$ s.t. $\forall b_1, b_2 >$ we have $|\int_{b_1}^{b_2} f(x)\,dx| < \epsilon$. Since $|f(x)| \leq g(x)$, we obtain $|\int_{b_1}^{b_2} f(x)\,dx| \leq \int_{b_1}^{b_2} |f(x)|\,dx \leq \int_{b_1}^{b_2} g(x)\,dx$. Then since $\int_a^{+\infty} g(x)\,dx$ exists, we can choose such that $\int^{+\infty} g(x)\,dx < \epsilon$.
Then for all $b_1, b_2 >$ we have $\int_{b_1}^{b_2} g(x)\,dx < \epsilon \implies |\int_{b_1}^{b_2} f(x)\,dx| < \epsilon$. Thus, $\int_a^{+\infty} f(x)\,dx$ exists.

5.6. Remark: all you need is $|f(x)| \leq g(x)$ for all $x >$ for some sufficiently large.

5.7. Example: $\int_1^{+\infty} \frac{\ln x}{x^2} dx$.

Notice $|\frac{\ln x}{x^2}| < \frac{x^{12}}{x^2} = x^{-2}$.